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ON THE AXIOMS OF PLASTICITY

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Abstract—In the present paper, some propositions usually considered as independent axioms in plasticity theory, including Il'yushin's postulate, are obtained as consequences of a general axiomatic structure of continuum damage mechanics. This fact indicates that continuum damage mechanics and plasticity theory have common conceptual foundations, notwithstanding that their phenomenological manifestations are different for many aspects. The formal framework used is able to consider different plasticity constitutive models and some of the non-classical situations discussed recently in technical literature on the basic concepts of plasticity theory, as the loss of convexity of the yield surface and the violation of Il'yushin's postulate in non-associative plasticity. A comparative analysis with other proposals on foundations of plasticity in technical literature is developed. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

An introductory attempt to clarify the foundations of the rich crop of models of damage evolution in solids has been recently formulated (Mariano and Augusti, 1997). For this purpose, the formalism developed by Noll (1972) and Coleman and Owen (1974, 1975, 1977) has been used in a non-properly classical style. Although much more must be done on the subject, the above-mentioned results permit one to prove some facts such as, for example, the meaning of damage potentials introduced by many authors [see Krajcinovic (1996); Lemaitre (1992); Lubarda and Krajcinovic (1995); Bazant and Pijauder-Cabot (1988)]. Also they permit one to point out the existence of basic analogies between the description of plasticity and dissipative evolution of damage, notwithstanding that they are deeply different for many physical aspects. The analysis of these analogies (and hence the attempt to attribute both theories—plasticity and damage mechanics—to the same metamodel, at least in part) is the main purpose of the present paper. The existence of such analogies in a particular case has recently been pointed out by Frantziskonis (1994). At the end of his paper, he proposed the following question: "Is it possible that a more general universal equivalence holds for all plasticity and damage theories?". Such an idea is, in the present paper, developed in a general sense. Some of the propositions which have been so far considered as axioms for a rational establishment of plasticity theory [see the detailed discussion in Lucchesi and Podio-Guidugli (1988, 1990); Lucchesi *et al.* (1992, 1993)] are derived as particular cases of general propositions in Mariano and Augusti (1979)†.

To explain the results in the following, a general abstract formalism is used. This choice is motivated by the fact that such results are general and, hence, they are independent from a specific model of plasticity, because they are referred to the metamodel of plasticity. Moreover, they can be specified to a particular constitutive model (or to a theory), by substituting to any mathematical object used its specific counterpart in the theory considered‡. The only restriction is that the considered theory must satisfy the axioms in Mariano and Augusti (1997). Such a constraint is not very restrictive. The axioms introduced for continuum damage mechanics (in addition to Noll's general ones of mechanics)

† This derivation, as presented in the following, is valid both for a global description of the body and for a single material point, while the above-mentioned propositions are referred to a single material point in the quoted technical literature.

‡ To this end some examples are also provided with the hope of elucidating clearly the sense in which certain mathematical objects are introduced.

are, in fact, simple physical properties of dissipative systems, which are tacitly assumed (at least in the writer's opinion) in the existing technical literature on foundations of plasticity. The only basic trick is the reference of some of the properties of state transformations to a remote horizon (Axiom 1), toward which subsequent yield surfaces (or loading surfaces) move when different plasticity (or damage) phenomena occur. Finally, it is possible to see that some of the recent discussions about the foundations of plasticity (Eve *et al.*, 1990a, 1990b; Lee, 1994; Lubarda *et al.*, 1996; Naghdi and Trapp, 1975) can find a natural place in the picture described here.

2. PRELIMINARY CONCEPTS AND DEFINITIONS

In this section, the framework presented in Mariano and Augusti (1997) is summarized briefly to make the paper self-consistent.

2.1. *States of the body, mechanical processes and actions*

Some concepts by Coleman and Owen (1975, 1977) are also summarized and their use in non-classical style is specified. These concepts are quoted because they constitute the formal fundamental elements necessary to the development of the above-mentioned general framework for continuum damage mechanics.

In the following, the state of the body is indicated by σ. With the work state, the n-plet of fields which describe the mechanical behavior of the body is denoted. In such a way, the subsequent developments are referred to a global description of the body. Most of the remarks discussed are also valid if σ is referred to a single material point of the body†, as in general usual in continuum mechanics. To fix ideas, in the case of simple elastic–plastic materials

$$\sigma \equiv (\mathbf{E}-\mathbf{E}_P, \mathbf{E}_P, k, \alpha) \tag{1}$$

where $\mathbf{E}$ and $\mathbf{E}_P$, k and α are the fields of Lagrangian measures of the total and plastic strain, the strain hardening and the back stress, respectively (Naghdi, 1990). Thus, σ can assume other meanings for different kinds of theories; for example:

$$\sigma \equiv (\mathbf{E}, \mathbf{D}, \theta) \tag{2}$$

in the case of brittle thermoelastic solids [see Krajcinovic (1996), and many references which can be found in it], where $\mathbf{D}$ is the damage tensor field and θ the absolute temperature field; or

$$\sigma \equiv (\mathbf{E}-\mathbf{E}_P, \mathbf{E}_P, \mathbf{D}, W) \tag{3}$$

for ductile elastic–plastic solids (Lubarda, 1994a), where W is a set of plastic internal variables different from $\mathbf{E}_P$. The state space is denoted by Σ and is assumed to have the elementary topological properties necessary to make sense of the operations that will be introduced in the following. Identical positions to eqns (1)–(3) can be made when σ is referred to a material point instead of the whole body. In such a case, the elements of σ are elements of a finite dimensional vectorial space‡. The interaction between the body and the external environment is described by a time-dependent operator indicated by P' (often by P for simplicity) acting on Σ during a finite time interval $[0, d_P]$ (d_P is the process duration). The process P can represent a load and/or deformation induced and/or temperature induced histories. The set of possible P is indicated by Π. The process P is considered here as an operator acting on those elements of the n-plet of fields σ that are observable. Internal variables are excluded from the action of P because their evolution is not directly influenced by the interaction of the body with the external environment. Indeed, this exclusion is a

† The reader can easily evaluate this fact from time to time.

‡ Most of the results in the following are referred to the global description of the body. When the results are restricted to single material point, such a case is explicitly underlined.

consequence of the fact that the rate of internal variables depends from observable variables and the same internal ones by an evolution phenomenological law and is not related directly with balance principia. In Noll (1972), the process P is the deformation process, while in Coleman and Owen (1975) it is an object that determines a continuum mapping ρ_P in the state space. The mapping ρ_P is the state-transformation induced by P. Such a mapping is defined as $\rho_P: D(P) \to R(P)$, where $D(P)$ and $R(P)$ are domain and range of the process P, respectively, and are subsets of Σ. $\rho_P\sigma$ represents, in the following, the final state for $t = d_P$; $\rho_{P'}\sigma$ is the path from σ to $\rho_P\sigma$ in the state space (i.e. the set of all states crossing in the state transformation induced by P); $\rho_{P'}\ {}_{'}\sigma$ the state of the body for $t = \bar{t} < d_P$.

In the formalism used in the present paper, a constitutive relation is a mapping from Σ into an appropriate functional space, which is subjected to invariance properties and has associated representation rules [see Williams (1980), for the detailed explanation of this latter fact]. For example, the stress map $\hat{\mathbf{S}}$ associated to every state a symmetric second-order tensor in the case of simple materials. In the case of microstructured continua (Capriz, 1989), $\hat{\mathbf{S}}$ relates every state σ with a pair $(\mathbf{S}, \mathfrak{s})$ in which $\mathbf{S}$ is a second-order tensor (the macroscopic stress tensor), that is in general non-symmetric†, and $\mathfrak{s}$ is the microscopic stress tensor related to the field additional to the displacement one. $\mathfrak{s}$ is an element of the cotangent bundle of the microstructural space (Augusti and Mariano, 1996; Makowski *et al.* 1980; Segev, 1994).

The following hypotheses are assumed about Σ: (I) there is at least one element $\bar{\sigma} \in \Sigma$, called base state, such that the set

$$\Pi\bar{\sigma} \stackrel{\text{def}}{=} \{\rho_P\bar{\sigma} \mid P \in \Pi, \bar{\sigma} \in D(P)\} \tag{4}$$

is dense in Σ. (II) In the set of compatible pairs of processes

$$\mathscr{P} \stackrel{\text{def}}{=} \{(P'', P') \in \Pi \times \Pi \mid D(P'') \cap R(P') \neq \varnothing\} \tag{5}$$

it is possible to define a map $(P'', P') \mapsto P''P'$ such that

$$D(P''P') = \rho_{P'}^{-1}(D(P'') \cap R(P')) \tag{6}$$

and $\forall \sigma \in D(P''P')$, $\rho_{P''P'}\sigma = \rho_{P''}\rho_{P'}\sigma$.

Let $\Pi \diamond \Sigma \stackrel{\text{def}}{=} \{(P, \sigma) \in \Pi \times \Sigma \mid \sigma \in D(P)\}$ be the set of compatible pairs. A function

$$a(\cdot, \cdot): \Pi \diamond \Sigma \to \Re \tag{7}$$

is called action if it satisfies the following properties: (i) if $(P'', P') \in \mathscr{P}$ and $\sigma \in D(P''P')$, then

$$a(P''P', \sigma) = a(P', \sigma) + a(P'', \rho_{P'}\sigma) \tag{8}$$

(ii) $a(P;): D(P) \to \Re$ is continuous for any $P \in \Pi$.

An action $a(\cdot, \cdot)$ has the conservation property at the state σ if its value is zero on all cycles at σ; while, if for any $\varepsilon > 0$ there is a neighborhood $\mathfrak{J}(\sigma)$ of σ such that, whenever $\rho_P\sigma \in (\sigma)$, $a(P, \sigma) > -\varepsilon$, then $a(\cdot, \cdot)$ is said to have the dissipation property at the state σ. As it can be easily shown, the action is an object that has the general properties of the work, or the Hamiltonian action. Examples of actions are

† The stress tensor is symmetric in many cases of structured continua. This fact depends on the choice of the microstructural descriptor and, hence, on the action of the infinitesimal generator of the action of the orthogonal group on the manifold in which the field additional to the displacement one takes values [see Capriz (1985, 1989)].

$$a(P,\sigma) = \int_0^{d_P} \int_{\mathscr{B}} \mathbf{T}_R : \dot{\mathbf{F}} \, dV \, dt + \int_0^{d_P} \int_{\mathscr{B}} \mathbf{b} \cdot \mathbf{v} \, dV \, dt \tag{9}$$

for simple bodies, where $\mathbf{T}_R$ is the stress tensor in the reference configuration, $\mathbf{F}$ the gradient of deformation, $\mathbf{b}$ the density of body forces, $\mathscr{B}$ a regular part of the body, or

$$a(P,\sigma) = \int_0^{d_P} \int_{\mathscr{B}} \mathscr{L}(\mathbf{x}, \dot{\mathbf{x}}. \nabla \mathbf{x}, \mathscr{N}, \dot{\mathscr{N}}, \nabla \mathscr{N}) \, dV \, dt \tag{10}$$

in the elastic dynamics of microcracked bodies (Mariano, 1996), where $\mathscr{L}$ is the Lagrange density, $\mathbf{x}$ the position vector, $\mathscr{N}$ the dipole approximation of the microcrack density function (Lubarda and Krajcinovic, 1993).

2.2. *Elements of the axiomatic framework of continuum damage mechanics*

In the subsequent developments the following symbols are adopted:

- With the letter $\mathscr{A}$ a proper subset of Σ is indicated. It is the set of admissible states of the body with respect to a given condition. Out of $\mathscr{A}$ the study of the mechanical behavior of the body does not make sense with respect to the mechanical typology considered. In $\mathscr{A}$, irreversible state-transformations can be defined. Moreover, subsets of $\mathscr{A}$, containing states which can be connected one another by elastic state transformations, can be individualized by yield or failure surfaces.
- Π^c is the set of processes inducing deformations.
- $\hat{P}$ indicates the process consisting in the product between unloading and relaxing of the material.
- If $\mathfrak{J}$ is a subset of Σ, the symbol $\{a_\sigma \to \mathfrak{J}\}$ indicates the set defined as

$$\{a_\sigma \to \mathfrak{J}\} \equiv \{a(P,\sigma) \mid P \in \Pi, \sigma \in D(P), \rho_P \sigma \in \mathfrak{J}\} \tag{11}$$

i.e. $\{a_\sigma \to \mathfrak{J}\}$ is the set of values of the action calculated along all paths starting from the initial states σ, varying P in Π (e.g. the values of the work expended going into $\mathfrak{J}$, starting from σ)†.

Axiom 1. (Closure of the admissible set of states): $\mathscr{A}$ is a proper non-empty subset of Σ, closed with respect to the topology in which the stress map $\hat{\mathbf{S}}(P,\sigma)$ is uniformly continuous with respect to the states‡.

Axiom 2. (Axiom of integrity): the set of all base states of Σ is in $\mathscr{A}$. Such base states are attainable in reversible manner from each other.

Axiom 3. (Inf-boundness of energy and dissipation to failure): two actions exist, $a'(\cdot,\cdot)$ conservative and $a''(\cdot,\cdot)$ dissipative, both at the base states; each of these actions is inf-bounded, i.e.

$$|\inf\{a'_\sigma \to \partial\mathscr{A}\}| < \infty \quad \text{and} \quad |\inf\{a''_\sigma \to \partial\mathscr{A}\}| < \infty. \tag{12}$$

In the following, P^{-1} indicates the inverse process, and P_{rel} the relaxation process of the body [see for rigorous definition Noll (1972)]. Also P^0 refers to either P^{-1} and P_{rel}.

Definition 1. The process $P \in \Pi$, starting from a given state $\sigma_0 \in \mathring{\mathscr{A}}$, is a damaging process with respect to the action $a(\cdot,\cdot)$, if:

† Note that $\{a_\sigma \to \mathfrak{J}\} \subseteq \Re$, being $\Re$ the set of real numbers.

‡ To permit a clear mental picture, $\mathscr{A}$ can be considered as the closure of a regularly open proper subset of the state space. In this case, if the boundary of $\mathscr{A}$ is chosen to coincide with the first plasticization (given for example by von Mises' yield criterion), the interior of $\mathscr{A}$ is the maximal elastic set related to such a criterion (see Corollary 5 below).

• the process and its reversal, after relaxation, are not cyclical, i.e.

$$(P,\sigma_0),\quad (P^{-1}P,\sigma_0),\quad (P_{\mathrm{rel}}P,\sigma_0)\notin(\Pi\diamond\Sigma)_{\mathrm{cycl}}\overset{\mathrm{def}}{=}\{(P,\sigma)\,|\,(P,\sigma)\in(\Pi\diamond\Sigma),\rho_P\sigma=\sigma\} \tag{13}$$

• the minimum amount of energy required for the body failure, from the final state of the process, is smaller than the initial state σ_0, i.e.

$$\inf\{a_{\sigma_0}\to\partial\mathscr{A}\}\geqslant\inf\{a_{\sigma_2}\to\partial\mathscr{A}\} \tag{14}$$

where $\sigma_2=\rho_{\mathrm{P}^{-1}\mathrm{P}}\sigma_0$, or $\sigma_2=\rho_{\mathrm{P}_{\mathrm{rel}}\mathrm{P}}\sigma_0$.

The equality sign holds when both "inf" are equal to zero, i.e. when the body is in an unstable critical state belonging to the boundary of $\mathscr{A}$.

The set of all damaging processes is indicated by Π^0 in the following.

Axiom 4. (Possibility of damage)†:

(i) $\forall\sigma\in\mathscr{A}$, $\exists P\in\Pi$ such that $\sigma\in D(P)$ and $\rho_{\mathrm{P}}\sigma\in\partial\mathscr{A}$;
(ii) $\forall\sigma\in\mathscr{A}$, if $\rho_{P|_{t=\bar{t}}}\sigma\in\mathscr{A}$ then $\rho_{P|_{t=\bar{t}}}\sigma\in\mathscr{A}$, $\forall\bar{t}\leqslant\bar{t}\leqslant d_{\mathrm{P}}$.

If P is such that $\rho_{\mathrm{P}}\sigma\in\partial\mathscr{A}$, then $P\in\Pi^0$.

Proposition 1. $\forall P\in\Pi^0$ and $\sigma_0\in\mathring{\mathscr{A}}$, $\forall a(\cdot,\cdot)$ for which P is a damaging process, if $a(\cdot,\cdot)$ is such that $\inf\{a_{\sigma_0}\to\partial\mathscr{A}\}$ and $\inf\{a_{\sigma_2}\to\partial\mathscr{A}\}$ exist and are finite, being $\sigma_2=\rho_{P^0P}\sigma_0$, $a(P^0P,\sigma_0)$ enjoys dissipative features with respect to process P^0P, i.e.

$$a(P^0P,\sigma_0)>0. \tag{15}$$

Definition 2. Consider two states in $\mathscr{A}$. σ and σ', σ' is said to be a damaged state with respect to σ if σ and σ' can be connected by at least one path induced by a process which causes damage; i.e. if there exists $P\in\Pi^0$ such that $\rho_{P^0P}\sigma=\sigma'$.

Axiom 5. If σ' is a damaged state with respect to σ, σ' is attainable from σ only by means of damaging processes such that $\rho_{P^0P}\sigma=\sigma'$.

Corollary 1. If $\mathrm{Inv}(\sigma)$ is the set defined as

$$\mathrm{Inv}(\sigma)\overset{\mathrm{def}}{=}\{\rho_P\sigma\,|\,P\in\Pi^0,\sigma\in D(P),\rho_P\sigma\in D(P^{-1})\text{ and }(P^{-1}P,\sigma)\in(\Pi\diamond\Sigma)_{\mathrm{cycl}}\} \tag{16}$$

i.e. the set of all states approachable from σ by a reversible process, then, $\forall\sigma'\in\mathrm{Inv}(\sigma)$

$$\Pi^0\sigma'=\Pi^0\sigma \tag{17}$$

where

$$\Pi^0\sigma\overset{\mathrm{def}}{=}\{\rho_P\sigma\,|\,P\in\Pi^0,\sigma\in D(P)\}. \tag{18}$$

3. ELASTIC–PLASTIC STATE TRANSFORMATIONS

As already anticipated in the introductive notes, in this section, some of the basic elements of plasticity theory are discussed in the picture of foundations of damage models that have been briefly summarized above. The results obtained are presented in the form of Corollaries of the general propositions in Mariano and Augusti (1997).

† This axiom excludes restoration procedures.

The present treatment of foundations of plasticity is also related to other classical general analyses (Moreau, 1976, 1977; Halphen and Nguyen, 1975; Eve *et al.*, 1990a, 1990b) as will be discussed in detail in the following.

Definition 3. A process P induces an elastic state transformation if:

(a) $P \in \Pi^c$	(i.e. the process P induces deformations);
(b) $\hat{P}P$ and $P^{-1}P \in (\Pi \diamond \Sigma)_{\mathrm{cycl}}$	(i.e. loading–unloading process is a cyclic process);
(c) $a''(\hat{P}P, \sigma) = 0\ \forall \sigma \in D(P)$	(i.e. the process P is not dissipative)†.

Definition 4. A process P induces an elastic–plastic state transformation if:

(a′) $P \in \Pi^c \cap \Pi^0$;

(b′) $\forall t \in [0, \mathrm{d_P}]$, every subset $\mathscr{C}$ of the body $\mathscr{B}$, mapped in a continuum arc by the reference placement $\ell^0 : \mathscr{B} \to \mathfrak{R}^3$ at the instant $t = 0$, is mapped also in a continuum arc in the actual placement;

(c′) if time intervals exist $(t_i, t_{i+1}) \subset [0, \mathrm{d_P}]$, $i = 1, 2, \ldots, n$, in which the segment $P'_{S_i} \notin \Pi^0$ (i.e. in which the part of the process related to such intervals does not produce irreversible deformations), with $t \in (t_i, t_{i+1})$, then at least one index i^* exists for which $P'_{S_{i^*}}$ induces an elastic state transformation;

(d′) the path $\rho_{\hat{P}'}(\rho_P \sigma)$ is elastic (i.e. elastic unloading).

In the subsequent developments, Π^{el} and Π^{ep} represent the sets of processes inducing elastic and elastic–plastic state transformations, respectively.

Definition 5. Given two states, σ and σ', σ' is said to be a plastic state with respect to σ if there exists $P \in \Pi^{\mathrm{ep}}$ such that $\rho_P \sigma = \sigma'$.

Corollary 2. States that are attainable from one another by a process which induces elastic state-transformations have the same set of possible plastic states.

Proof. By Definitions 3 and 4, $P \in \Pi^{\mathrm{ep}} \Rightarrow P \in \Pi^0$ and $P \in \Pi^{\mathrm{el}} \Rightarrow \rho_P \sigma \in \mathrm{Inv}(\sigma)$, being $\mathrm{Inv}(\sigma)$ the set of attainable states starting from σ by reversible state transformations. As a consequence, Corollary 2 is a particular case of Corollary 1.

Corollary 2 contains, in substance, both the axioms of invariance of the elastic range under elastic continuations‡ and the axiom of insensitivity of plastic stretching to elastic cycles§, that are used in the theory of materials with elastic range (Lucchesi and Podio-Guidugli, 1988, 1990). Indeed, a trivial consequence of Corollary 2 is also the invariance of every measure of plastic deformations under elastic and rigid-body processes.

In analogous way to damage, it is possible to individualize the minimum finite quantity of dissipation necessary to induce plastic deformations. Such a quantity is indicated by $\Lambda_\sigma^{\mathrm{ep}}$.

Formally, if there exists at least one process $P \in \Pi^{\mathrm{ep}}$ such that $\sigma \in D(P)$, for every σ, the quantity $\Lambda_\sigma^{\mathrm{ep}}$ (lower bound of plasticity) is defined as

$$\Lambda_\sigma^{\mathrm{ep}} \equiv \inf_{\hat{P}P,\, P \in \Pi^{\mathrm{ep}}} \{a''_\sigma \to \mathscr{A}\}. \tag{19}$$

This definition is proposed in analogy with the one of lower bound of damage Λ_σ given in Mariano and Augusti (1997), where

† As a matter of fact, point (c) in Definition 3 can be neglected if pseudoelastic behavior is considered. In this case a'' is greater than zero, but its value on cycles is less than the lower bound of plasticity as introduced in the following.

‡ This axiom affirms that every deformation history, whose initial value is a rigid-body rotation, is such that the associated elastic region is equal to the one of every elastic continuation of it.

§ This axiom affirms that the symmetric part of the plastic velocity gradient is invariant under elastic cycles.

$$\Lambda_\sigma \equiv \inf_{P^\circ P, P \in \Pi^0} \{a''_\sigma \to \mathscr{A}\}. \tag{20}$$

It is obvious that both Λ_σ and Λ_σ^{ep} can be referred to a generic action. Previous definitions have been proposed by using $a''(\cdot,\cdot)$ only to give a better physical meaning†.

By using Proposition 1, Axiom 3 and considering that $\Pi^{ep} \subseteq \Pi^0$, it is trivial to note that

$$+\infty > \Lambda_\sigma^{ep} \geqslant \Lambda_\sigma \geqslant 0. \tag{21}$$

To give a physical sense to the mathematical developments in the following, it is necessary to refer them to cases in which $\Lambda_\sigma^{ep} > 0$. This proposition means two different things:

- the state σ is such that $\Lambda_\sigma^{ep} > 0$ whatever is the process $P \in \Pi^{ep}$ with finite duration (constitutive restriction);
- only a subset $\mathfrak{B}$ of Π^{ep} such that $\Lambda_\sigma^{ep} > 0$ at the state σ, being Λ_σ^{ep} calculated on $\mathfrak{B}$ is considered (restriction on processes).

When in subsequent developments $\Lambda_\sigma^{ep} > 0$ is assumed, one of the two previous cases is tacitly considered.

Corollary 3. Let $a(\cdot,\cdot)$ be an inf-bounded action. If $\Lambda_\sigma^{ep} > 0$ and there exists an instant t' such that the segment P_S^t, $t \in [0, t']$, of a process $P \in \Pi^{ep}$ implies $a(P_S^t, \sigma) \geqslant 0$, the elastic–plastic process starts by an elastic segment (i.e. $P \in \Pi^{ep} \Rightarrow P = P''P'$, with $P' \in \Pi^{el}$ and $P'' \in \Pi^{ep}$)‡.

Proof. Because $a(P,\cdot)$ is a continuous function and $a(P^{t=0},\cdot) = 0$, a time interval exists $[0, t''] \subset [0, d_P]$ in which the segment P_S^t is such that $a(P_S^t, \sigma) < \Lambda_\sigma^{ep}$, $\forall t \in [0, t'']$. As consequence, if $t \in [0, t'] \cap [0, t'']$, $P_S^t \notin \Pi^0$, $P_S^t \in \Pi^c$ and $\rho_{P_S^t}\sigma \in \mathrm{Inv}(\sigma)$. By Definition 4, $P_S^t \in \Pi^{el}$.

A strong version of Corollary 3 can be obtained if the point (c′) of elastic–plastic state transformation is replaced as follows: (c″) if there exist time intervals $(t_i, t_{i+1}) \subset [0, d_P]$, $i = 1, 2, \ldots, n$, in which the segment $P_{S_i}^t \notin \Pi^0$, $t \in (t_i, t_{i+1})$, then $P_{S_i}^t$ induces elastic state transformations.

Corollary 3*bis*. If at a given state σ there is $\Lambda_\sigma^{ep} > 0$, all elastic–plastic processes start from σ by an elastic segment.

In the Owen's mechanical theory of materials with elastic range (Owen, 1970), the result in Corollary 3bis is a direct consequence of the definition of elastic region.

In Definition 3, point (c′) is considered instead of point (c″) in order to take into account non-associative elastic–plastic state transformations, which can be associated to a process that violates Il'yushin's postulate already in the hardening regime. However, such a violation does not necessarily imply that the material becomes unstable (Lubarda *et al.*, 1996). Such situations occur in geomaterials.

Corollary 4. Let σ be a state in which a material admits elastic–plastic behavior. At least one process P (with $\sigma \in D(P)$) for which at least one measure of plastic deformation is different from zero exists.

Corollary 4 is obvious. Indeed, if Λ_σ^{ep} can be defined (the material admits elastic–plastic behavior), at least one process $P \in \Pi^{ep}$ such that $\sigma \in D(P)$ exists and $\Lambda_{\rho_P \sigma}^{ep} \geqslant 0$. Corollary 4 contains the axiom of history continuations with plastic stretching§ (Lucchesi and Podio-Guidugli, 1990).

† $a''(\cdot,\cdot)$, in fact, is a dissipative action.

‡ An analogous statement is valid when in the elastic region the material is pseudoelastic.

§ This axiom affirms that, if at the end of a deformative process the state is on the boundary of the elastic region, at least one elastic–plastic continuation exists.

Corollary 5. In hypotheses of Corollary 3, for every state $\sigma \in \mathscr{A}$, such that $\Lambda_\sigma^{ep} > 0$, a maximal elastic region $El(\sigma)$ exists.

Proof. By exclusion of rigid body motions, $\forall \sigma' \in \mathrm{Inv}(\sigma)$, $\inf\{a''_\sigma \to \sigma'\} \underset{\mathrm{def}}{\leqslant} \Lambda_\sigma^{ep}$ [see Theorem 4 in Mariano and Augusti (1997)]. Consequently, because of $\Pi^{el}\sigma \overset{\mathrm{def}}{=} \{\rho_P\sigma \mid P \in \Pi^{el}, \sigma \in D(P)\} \subset \mathrm{Inv}(\sigma)$, the maximal elastic set is defined by the following expression:

$$\hat{E}l(\sigma) \overset{\mathrm{def}}{=} \left\{\sigma' \mid \sigma' \in \Pi^{el}\sigma, \inf_{P \in \Pi^{ep}} \{a''_\sigma \to \rho_P\sigma'\} < \Lambda_\sigma^{ep}\right\}. \tag{22}$$

Alternative proof. If $\Lambda_\sigma^{ep} > 0$, $\Pi^{el}\sigma$ is not empty by Corollary 3. Moreover, starting from σ and moving into the elastic region, it is possible to do cyclical deformative processes $\hat{P}P$ such that $0 \leqslant a''(\hat{P}P, \sigma) < \Lambda_\sigma^{ep}$. The inequality sign in the l.h.s. is referred to the possibility that the body admits pseudoelastic behavior.

This Corollary coincides with the axiom of existence of a maximal elastic set (Lucchesi and Podio-Guidugli, 1988).

Corollary 6. Every subset $\tilde{\Pi}^{el} \subset \Pi^{el}$ such that $\tilde{\Pi}^{el}\sigma \subseteq El(\sigma)$ and the set of all possible paths (i.e. state transformation) are arcwise connected.

This Corollary is a direct consequence of the fact that $El(\sigma)$ is a set of states which are attainable each other. It is a direct consequence of the conservative nature of elastic state transformations.

Let $P \in \Pi^0$ be a differentiable function of the time. Consider also P belonging to Lin $\mathfrak{F}$, where $\mathfrak{F}$ is an inner product vectorial space (Noll, 1972). In other words, P is considered in the following as a deformation induced history $\mathbf{F}(t)$. Consider the functional

$$\tilde{a}(\hat{P}P, \sigma) = \int_0^{d_P} \int_{\mathscr{B}} \mathscr{K}(\mathrm{S}(\sigma), \hat{P}P) \cdot D_{\hat{P}P}\, \mathrm{d}V\, \mathrm{d}t. \tag{23}$$

The following assumptions hold:

- $\mathscr{K}(\cdot, \hat{P}P)$ is a continuum mapping of the stress mapping $S(\cdot)$;
- $D_{\hat{P}P} = \frac{1}{2}(\tilde{P}\tilde{P} - \tilde{P}^T\tilde{P})$, with $\tilde{P} = \hat{P}P$;
- $\int_{\mathscr{B}} \mathscr{K}(\mathrm{S}(\sigma), \hat{P}P) \cdot D_{\hat{P}P}\, \mathrm{d}V \overset{\mathrm{def}}{=} f(t)$ is a scalar function of the time.

Under these assumptions, $\tilde{a}(\cdot,\cdot)$ is an action. Moreover, as a consequence of Proposition 1, the Corollary in the following holds:

Corollary 7. If $\tilde{a}(\cdot,\cdot)$ is inf-bounded, then $\forall P \in \Pi^{ep}$

$$\int_0^{d_P} \int_{\mathscr{B}} \mathscr{K}(S(\sigma), \hat{P}P) \cdot D_{\hat{P}P}\, \mathrm{d}V\, \mathrm{d}t \geqslant 0. \tag{24}$$

If the equality sign is valid, the path $\rho_{(\hat{P}P)}\sigma$ is an unloaded path.

A path $\rho_{P'}\sigma$ is unloaded if $S(\rho_{P'\bar{t}}\sigma) = 0$, $\forall \bar{t} \in [0, \mathrm{d_P}]$, and $\mathscr{K}(S(\rho_{P'=\bar{t}}\sigma), P) = 0$. The mapping $\mathscr{K}(\cdot,\cdot)$ must be interpreted as the Kirchhoff stress tensor. If the state σ is not referred to the whole body, but to a material point, Corollary 7 can be expressed in the following way:

Corollary 7bis. If $\tilde{a}(\cdot,\cdot)$ is inf-bounded, then $\forall P \in \Pi^{ep}$

$$\int_0^{d_P} \mathscr{K}(S(\sigma), P^0 P) \cdot D_{P^0 P} \, dt \geqslant 0. \tag{25}$$

Corollaries 7 and 7bis are different versions of Il'yushin's postulate and are valid in a context in which the stability of the material is not required always for all process segments. Equation (24) [as eqn (25)] does not exclude that Il'yushin's postulate is violated during a generic segment of the process considered. Such a violation occurs in the case of geomaterials which present pressure dependence, plastic volumetric changes, frictional effects and admit non associative flow rules as sands that dilate under shear (as pointed out by Lade *et al.*, 1987, by using triaxial tests). Moreover, the loss of Il'yushin's condition is not necessarily related to the loss of stability (Lade *et al.*, 1987; Lubarda *et al.*, 1996; Maier, 1970). On the other hand, eqn (24) is a weak formulation of Il'yushin's postulate with respect to eqn (25). Indeed, eqn (24) does not exclude a violation of Il'yushin's postulate in a proper subset $\mathscr{B}'$ of the body $\mathscr{B}$, even during the entire process duration $d_{P^0 P}$. When instability occurs, such a typology of violation can correspond to formation of shear bands (Ottosen and Runesson, 1991, Neilsen and Schreyer, 1993, Rudnicki and Rice, 1975, Yamamoto, 1978).

In its original formulation, Il'yushin's postulate is associated to cycles of strain. These processes are cyclical only in the sense of total deformation. On the contrary, to a cyclical total deformation does not correspond a cycle in the state space, as underlined by means of Corollaries 7 and 7bis that are referred to a $\hat{P}P$ process, which does not implicate a cyclical state transformation.

If $\mathbf{D}_{(P^*P)'}(\mathbf{x}) = \dot{\mathbf{E}}(\mathbf{x}, t)$ (being $\dot{\mathbf{E}}(\mathbf{x}, t)$ the time rate of the deformation tensor), Corollaries 7 and 7bis reduce to Naghdi and Trapp's inequality (Naghdi and Trapp, 1975; Srinivasa, 1997).

In non-isothermal case, for deformation–temperatures cycles, consider the action given by

$$\hat{a}(P, \sigma) \equiv \int_0^{d_P} \int_{\mathscr{B}} \left(\frac{J}{\rho_0 \vartheta} \mathscr{K}(\sigma) \cdot \mathbf{L} - \frac{\varepsilon}{\vartheta^2} \dot{\vartheta} \right) dV \, dt \tag{26}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ (being $\mathbf{F}(t)$ the deformation gradient), $J = \det \mathbf{F}$, ϑ is the temperature, ε the internal energy, ρ_0 the reference mass density. With reference to the action in eqn (26), the process $\hat{P}P$ considered is given by the pair $(\mathbf{L}, \dot{\vartheta})$.

As a consequence of Proposition 1, if the process P, considered for the action (26), induces elastic–plastic state transformations, the two corollaries in the following hold.

Corollary 8. If the action $\hat{a}(P, \sigma)$ is inf-bounded, then $\forall P \in \Pi^{ep}$

$$\int_0^{d_P} \int_{\mathscr{B}} \left(\frac{J}{\rho_0 \vartheta} \mathscr{K}(\sigma) \cdot \mathbf{L} - \frac{\varepsilon}{\vartheta^2} \dot{\vartheta} \right) dV \, dt \geqslant 0. \tag{27}$$

Corollary 8*bis*. If the action

$$\hat{a}_1(P, \sigma) \equiv \int_0^{d_P} \left(\frac{J}{\rho_0 \vartheta} \mathscr{K}(\sigma) \cdot \mathbf{L} - \frac{\varepsilon}{\vartheta^2} \dot{\vartheta} \right) dt \tag{28}$$

is inf-bounded, then $\forall P \in \Pi^{ep}$

$$\int_0^{d_P} \left(\frac{J}{\rho_0 \vartheta} \mathscr{K}(\sigma) \cdot \mathbf{L} - \frac{\varepsilon}{\vartheta^2} \dot{\vartheta} \right) dt \geqslant 0. \tag{29}$$

The version of Il'yushin's postulate in Corollary 9bis has been proposed by Lucchesi and Silhavy (1991). Corollary 8 is its weak version in the sense discussed in the case of Corollary 7. In the above explained way, because of the generality of Proposition 1, it is possible to express generalized forms of Il'yushin's postulate even in case of plasticity theories of microstructured continua (in multifield description) and other cases of non simple materials†. As a consequence, it could be possible to discuss convexity of the stress range of macro- and micro-stresses even in the case of such continua because of the generality of Proposition 1.

Finally, note that $\Lambda_\sigma^{\mathrm{ep}}$ can be defined even with respect to a restricted class of processes (constitutive restriction on processes) as follows:

$$\hat{\Lambda}_\sigma^{\mathrm{ep}} \equiv \inf_{\hat{P}P,P\in\hat{\Pi}^{\mathrm{ep}}\subseteq\Pi^{\mathrm{ep}}} \{a''_\sigma \to \mathscr{A}\} \tag{30}$$

depending on the case under analysis.

4. SOME REMARKS ON YIELD SURFACES

The set

$$\Upsilon(\sigma) \overset{\mathrm{def}}{=} \{\sigma' \mid \inf\{a_\sigma \to \sigma'\} = \Lambda_\sigma^{\mathrm{ep}} > 0\} \tag{31}$$

can be considered as the admissibility surface connected to the yield surface related to σ. A more general expression than the previous one can be given by

$$\hat{g}^\sigma = g\left(\inf_{P\in\hat{\Pi}^{\mathrm{ep}}\subseteq\Pi^{\mathrm{ep}}} \{a_{\sigma'\in E1(\sigma)} \to \mathscr{A}\}\right) = 0 \tag{32}$$

where $g(\cdot)$ is a suitable smooth function. It can be placed in the stress space or strain space depending on the description chosen for σ. Various discussions are presented in technical literature about the choice of the space in which the yield surface must be considered [see Naghdi (1990); Lubarda (1994b); and references quoted therein].

Classically, given a state σ, the related yield surface is considered as a closed ipersurface without boundary which induces a partition in $\mathscr{A}$. It circumscribes the maximal elastic set.

If at a given state σ conditions of convexity of the stress range associated to the elastic region hold, classical analyses, developed by the help of convex analysis, are compatible with the picture described in the present paper, especially in the case of validity of eqn (25) (Eve *et al.*, 1990a, 1990b; Halphen and Nguyen, 1975; Hill, 1987; Lee, 1994; Moreau, 1976). On the other hand, if convexity is violated, analyses such as the ones proposed by Kim and Oden (1984, 1985) with the auxilium of generalized lower differentials for non convex and non-differentiable functions can be followed.

As shown by many authors (Naghdi and Trapp, 1975; Lucchesi and Podio-Guidugli, 1990; Lucchesi and Silhavy, 1991; Srinivasa, 1997), convexity of the stress range associated to the elastic region and Il'yushin's postulate are strictly connected. This relation arises naturally in the case in which the attention is focused on the state space of a single material point. In this case, this space can be considered as a subset (not necessarily proper) of a finite dimensional topological vector space and, hence, the convexity of the stress range follows from the work inequality as stated in Corollaries 7bis and 8bis by adopting classical arguments of convex analysis.

Moreover, taking into account the partition of $\mathscr{A}$ which can be made by using different yield surfaces, the arguments by Del Piero (1975) prove the existence and uniqueness of the solution of the evolution problem.

† Note that in the different versions of Il'yushin's postulate, that have been proposed, it is assumed that the processes considered are of Π^0-type with reference to the action considered.

$$\dot{\rho}_{P^t}\sigma = K(\rho_{P^t}\sigma, \dot{P}^t) \tag{33}$$

for elastic–plastic materials under special assumptions on K. Indeed, because of the properties of state transformations, eqn (27) is associated to the existence of a semigroup for field equations in physical space.

In the neighborhood of elements of $\Upsilon(\sigma)$ or $\hat{g}^{\sigma}$ (i.e. in a neighborhood of the first plasticization with respect to the state σ), when in the linearized case the state space can be considered to have Hilbert structure, two different considerations hold:

- The additive decomposition of total strain implicates the validity of Drucker's postulate of path independence in the small [cf Lee (1994)], when $\Lambda_{\sigma}^{ep} > 0$.
- In the case of convexity of $El(\sigma)$, given σ' such that $\inf\{a_{\sigma} \to \sigma'\} > \Lambda_{\sigma}^{ep}$, assume that a new convex set $El(\sigma')$ is associated to σ'. Hence, in a neighborhood of σ the evolution problem associated to $\rho_{P^t}\sigma$, varying t, can be considered as the motion of a convex set in a Hilbert space and analyzed with the help of techniques proposed by Moreau (1977).

5. CONCLUSIONS

A deduction of some propositions considered as axioms in plasticity theory has been developed within the limits of a general, physically consistent, axiomatic framework for continuum damage mechanics. Five well known axioms (invariance of the elastic range under elastic continuation, insensitivity of plastic stretching to elastic cycles, history continuations with plastic stretching, existence of a maximal elastic set, Il'yushin's postulate) has been obtained as corollaries of the general propositions in Mariano and Augusti (1997). In particular, Il'yushin's postulate has been placed in a context in which the stability of the material is not required for all processes with respect to all material points.

The analysis contains cases of non associative behavior and is valid also for plasticity theories in which the state of the body is described by a multifield approach (Lippmann, 1969; Le and Stumpf, 1996; Steinmann, 1996; Vardoulakis and Frantziskonis, 1992) or a non-local one (Eringen, 1981). Comparisons with classical works on the basic framework of plasticity have been shown. It appears possible to conclude that damage mechanics and plasticity theory have common conceptual foundations.

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